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MATTER COUPLING IN $N = 4$ SUPERGRAVITY

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An arbitrary number of abelian vector multiplets is coupled to $N = 4$ supergravity. The resulting action is invariant under global $SO(n, 6)$, where n is the number of vector multiplets, and under local $SU(4) \times U(1)$ transformations. The scalar fields of the theory parametrize the manifold $[SO(n, 6)/SO(n) \times SO(6)] \times [SU(1, 1)/U(1)]$. The role of the matter fields of the $N = 4$ Weyl multiplet in the Poincaré supergravity theory is clarified.

1. Introduction

Recently there has been much interest in unifying supergravity with the electro-weak and strong interactions (for recent reviews see [1]). Much of this work is based on the results of Cremmer et al. [2], in which the most general coupling of scalar and Yang-Mills supermultiplets to $N = 1$ supergravity has been constructed. These $N = 1$ models are not consistent at the quantum level and should be interpreted as effective actions only, based (presumably) on extended supergravity theories. The generalization to $N > 1$ is therefore of interest. Recently results have been obtained for the $N = 2$ theory [3, 4]. In this paper we investigate the coupling of abelian vector multiplets to $N = 4$ supergravity [5, 6].

Interest in $N = 4$ supergravity in this context has, among other things, been raised by the so-called no-scale models [7]. In these models the coupling of matter multiplets is chosen such that the scalar potential vanishes. It turns out that this phenomenon is associated with the presence of a non-compact symmetry group, which contains $SU(1, 1)$. These flat potentials are obtained in certain couplings to $N = 1$ and $N = 2$ supergravity [8–10], but it is well-known that the $SU(1, 1)$ symmetry occurs in a natural way in $N = 4$ supergravity [6]. It is therefore interesting to develop systematically the matter coupling for $N = 4$, and to establish a relationship between the lower N results and $N = 4$ supergravity.

The only matter multiplets in $N = 4$ supersymmetry (i.e. multiplets with physical states of helicity ≤ 1 only) are the vector multiplets [11]. They contain a vector gauge field which is an $SU(4)$ singlet, and spin- $\frac{1}{2}$ and scalar fields which transform under the fundamental and the 6-dimensional representation of $SU(4)$, respectively. Besides their $SU(4)$ assignment, these fields may take values in the algebra of an

arbitrary Lie-group ($N = 4$ Yang-Mills theory). In this paper we consider only the case where this Lie group is abelian.

Calculations in $N = 4$ supergravity are hampered by the absence of a complete tensor calculus for $N = 4$ multiplets, this in contradistinction to the $N = 1$ [12] and $N = 2$ [13] cases. Thus it would seem that the investigation of matter couplings necessitates the use of the Poincaré supergravity Noether procedure, and in this case such a calculation should certainly prove quite formidable (see [14], where a comparable calculation was done for $N = 2$). However, the off-shell $N = 4$ Weyl multiplet is known [15]. Even in the absence of auxiliary fields for the $N = 4$ matter multiplet, this allows the use of powerful superconformal methods [16] and leads to an intermediate tensor calculus for $N = 4$.

The $N = 4$ Weyl multiplet allows global $SU(1, 1)$ and local $U(1)$ transformations, associated to the scalar fields of the multiplet [15], which parametrize the $SU(1, 1)/U(1)$ coset space. A basic ingredient in our calculation is the assumption that these symmetries persist as symmetries of the equations of motion in the Poincaré theory coupled to matter, as they do in pure $N = 4$ Poincaré supergravity [6]. This assumption greatly simplifies all calculations, and its validity can be easily verified a posteriori.

Previous results on matter coupling in $N = 4$ supergravity have come from $N = 1$ supergravity in ten dimensions [17, 18]. On reduction to $d = 4$ this gives $N = 4$ supergravity coupled to six vector multiplets [17]. However, the result is hard to generalize to an arbitrary number of vector multiplets, since supergravity and matter fields are difficult to disentangle. The same difficulty arises when supergravity coupled to matter [18, 19] is reduced from $d = 10$ to $d = 4$.

In sects. 2 and 3 we discuss the coupling of a single vector multiplet to $N = 4$ conformal supergravity. The superconformal algebra is imposed on the matter fields. It closes modulo the equations of motion, when evaluated on the matter fields. From these equations of motion the full lagrangian is constructed. In sect. 4 we discuss the transition to the Poincaré gauge, and the elimination of auxiliary fields. There are two sets of scalar fields in the resulting theory. The fields ϕ_a of the Weyl multiplet parametrize $SU(1, 1)/U(1)$, and are physical fields of Poincaré supergravity. The scalar fields from the n vector multiplets correspond to $SO(n, 6)/SO(n) \times SO(6)$, as mentioned in [9]. As we consider at present only abelian vector multiplets which are minimally coupled there is no scalar potential. Generalizations which would include such potentials are briefly mentioned in sect. 5.

2. Matter coupling and duality invariance

The $N = 4$ vector multiplet [11] contains a vector field A_μ , spin- $\frac{1}{2}$ fields ψ_i and scalars ϕ_{ij} , $\phi_{ji} = -\phi_{ij}$ ($i, j = 1, \dots, 4$). The fields satisfy the conditions

$$\begin{aligned}\psi_i &= -\gamma_5 \psi_i, & \psi^i &\equiv (\psi_i)^* = \gamma_5 \psi^i, \\ \phi^{ij} &\equiv (\phi_{ij})^* = -\frac{1}{2} \epsilon^{ijkl} \phi_{kl}.\end{aligned}\tag{2.1}$$

Under infinitesimal SU(4) transformations they transform as

$$\delta\psi^i = \Lambda_j^i \psi^j, \quad \delta\phi^{ij} = \Lambda^{[i}{}_k \phi^{kj]}, \quad (2.2)$$

where the parameter Λ_j^i satisfies

$$\Lambda_i^j \equiv (\Lambda_j^i)^* = -\Lambda^j{}_i, \quad \Lambda^i{}_i = 0. \quad (2.3)$$

The lagrangian density of the $N = 4$ supersymmetric Maxwell theory reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^+ F_{\mu\nu}^+ - \frac{1}{2}\bar{\psi}^i \not{\partial} \psi_i - \frac{1}{4}(\partial_\mu \phi_{ij})(\partial_\mu \phi^{ij}) + \text{h.c.} \quad (2.4)$$

Here $F_{\mu\nu}^\pm$ is the (anti-)selfdual part of the field strength $F_{\mu\nu}$, i.e.

$$F_{\mu\nu}^\pm = \frac{1}{2}(F_{\mu\nu} \pm \frac{1}{2}\epsilon_{\mu\nu\lambda\rho} F^{\lambda\rho}), \quad (F_{\mu\nu}^\pm)^* = F_{\mu\nu}^\mp. \quad (2.5)$$

The action (2.4) is invariant under the rigid supersymmetry transformations

$$\begin{aligned} \delta A_\mu &= \bar{\epsilon}^i \gamma_\mu \psi_i + \text{h.c.}, \\ \delta \psi_i &= -\sigma_{\mu\nu} \epsilon_i F_{\mu\nu}^+ - 2 \not{\partial} \phi_{ij} \epsilon^j, \\ \delta \phi_{ij} &= \bar{\epsilon}_{[i} \psi_{j]} - \epsilon_{ijk} \bar{\epsilon}^k \psi^j. \end{aligned} \quad (2.6)$$

The algebra of supersymmetry transformations closes modulo gauge transformations of the vector field A_μ , and terms proportional to the equation of motion of the spinor ψ_i .

The lowest-order terms in the coupling of (2.4) to conformal supergravity were obtained in [15]. In fact, these Noether coupling terms were instrumental in the construction of the superconformal gauge multiplet itself. The full nonlinear superconformal multiplet was obtained in a form which allows noncompact SU(1,1) transformations, and has an associated local U(1) symmetry. Furthermore, the superconformal gauge algebra contains local SU(4) transformations. Since $N = 4$ Poincaré supergravity can be formulated with global $SU(4) \times SU(1,1)$ symmetry of the equations of motion, and local U(4) symmetry [20], it is clear that these superconformal symmetries should be preserved in the coupling to supersymmetric matter. In fact, we will assume that the SU(1,1) symmetry is realized as a symmetry of the equations of motion of the vector field A_μ . With this assumption the structure of terms in the action and transformation rules that contain $F_{\mu\nu}$ is determined completely.

To start we must fix the properties of the matter fields under local U(1) and Weyl transformations (dilations). The condition (2.1) and the reality of A_μ forces us to choose the chiral weights

$$c(A_\mu) = c(\phi_{ij}) = 0, \quad c(\psi_i) = -\frac{1}{2}. \quad (2.7)$$

Note that δA_μ , and the first term in $\delta\psi_i$, (2.6), appear to be inconsistent with this choice. As we shall see below, modifications of (2.6) which are implied by SU(1,1) invariance solve this problem. The weights under Weyl transformations are

$$w(A_\mu) = 0, \quad w(\phi_{ij}) = 1, \quad w(\psi_i) = \frac{3}{2}. \quad (2.8)$$

With these weights the action (2.4), coupled to conformal gravity with a vierbein e_μ^a with $w(e_\mu^a) = -1$, will be Weyl invariant.

In the remainder of this section we shall consider the implications of SU(1,1) symmetry. Our method is similar to that employed in the construction of $N = 8$ supergravity [21]. The superconformal gauge multiplet contains scalar fields ϕ_α , ($\alpha = 1, 2$) which have chiral weight $c(\phi_\alpha) = -1$, and transform as a doublet under SU(1,1). They satisfy the SU(1,1) \times U(1) invariant condition

$$\phi^\alpha \phi_\alpha = 1, \quad (2.9)$$

where $\phi^1 = (\phi_1)^*$, $\phi^2 = -(\phi_2)^*$. With these fields we can parametrize an SU(1,1) element U :

$$U = \begin{bmatrix} \phi_1 & -\phi^2 \\ \phi_2 & \phi^1 \end{bmatrix}, \quad (2.10)$$

which transforms under SU(1,1) \times U(1) as

$$U(x) \rightarrow CU(x)\Omega(x), \quad (2.11)$$

where $C \in$ SU(1,1), and

$$\Omega(x) = \begin{bmatrix} e^{-i\Lambda(x)} & 0 \\ 0 & e^{i\Lambda(x)} \end{bmatrix}. \quad (2.12)$$

On the other hand SU(1,1) transformations can act on $F_{\mu\nu}^+$ and

$$G_{\mu\nu}^+ \equiv -\frac{2}{e} \frac{\delta \mathcal{L}}{\delta F_{\mu\nu}^+} \quad (2.13)$$

in such a way that the equation of motion and the Bianchi identity, which read

respectively

$$\begin{aligned}\partial_\mu (e (G^{+\mu\nu} + G^{-\mu\nu})) &= 0, \\ \partial_\mu (e (F^{+\mu\nu} - F^{-\mu\nu})) &= 0,\end{aligned}\tag{2.14}$$

are preserved. Here e is the vierbein determinant, in anticipation of the coupling to (conformal) gravity. $SU(1,1)$ transformations act on the combinations

$$F_{1\mu\nu}^+ = \frac{1}{2}(G_{\mu\nu}^+ - F_{\mu\nu}^+), \quad F_{2\mu\nu}^+ = \frac{1}{2}(G_{\mu\nu}^+ + F_{\mu\nu}^+),\tag{2.15}$$

as

$$\begin{pmatrix} F_{1\mu\nu}^+ \\ F_{2\mu\nu}^+ \end{pmatrix} \rightarrow C \begin{pmatrix} F_{1\mu\nu}^+ \\ F_{2\mu\nu}^+ \end{pmatrix}.\tag{2.16}$$

One easily verifies that (2.14), and a normalization condition, restrict C to $SU(1,1)$.

The relation between $G_{\mu\nu}^+$ and $F_{\mu\nu}^+$ implied by (2.13) can only be preserved if $F_{\mu\nu}^+$ is coupled to the scalars (2.10). We identify the $SU(1,1)$ transformations in (2.11) and (2.16), so that the following $SU(1,1)$ invariant combinations can be formed:

$$U^{-1} \begin{pmatrix} F_{1\mu\nu}^+ \\ F_{2\mu\nu}^+ \end{pmatrix} = \begin{pmatrix} \bar{G}_{\mu\nu}^+ \\ \bar{F}_{\mu\nu}^+ \end{pmatrix}.\tag{2.17}$$

The action cannot be $SU(1,1)$ invariant. Its generic form [22] reads in the present case (terms related to the Maxwell field only)

$$\mathcal{L}_1 = -\frac{1}{4}e F_{\mu\nu}^+ G^{\mu\nu+} - \frac{1}{2}e \bar{F}_{\mu\nu}^+ \bar{G}^{\mu\nu+} + \text{h.c.}\tag{2.18}$$

The first term is the only allowed (and required) $SU(1,1)$ non-invariant term [22], the second is the only $SU(1,1) \times U(1)$ invariant combination of $\bar{F}_{\mu\nu}^+$ and $\bar{G}_{\mu\nu}^+$. Now we use the definition (2.13), and the explicit forms (2.15), (2.17) to conclude that

$$G_{\mu\nu}^+ = \frac{\phi^1 - \phi^2}{\phi^1 + \phi^2} F_{\mu\nu}^+ + \frac{2}{\phi^1 + \phi^2} H_{\mu\nu}^+, \tag{2.19a}$$

$$\bar{G}_{\mu\nu}^+ = H_{\mu\nu}^+, \tag{2.19b}$$

$$\bar{F}_{\mu\nu}^+ = \frac{1}{\phi^1 + \phi^2} F_{\mu\nu}^+ + \frac{\phi_1 - \phi_2}{\phi^1 + \phi^2} H_{\mu\nu}^+, \tag{2.19c}$$

where $H_{\mu\nu}^+$ is at this point undetermined, but independent of $F_{\mu\nu}^+$. Since $SU(1,1)$ and supersymmetry commute, it is the $SU(1,1)$ -invariant combination $\bar{F}_{\mu\nu}^+$ which must appear in the transformation rule of ψ_i (2.6). The chiral weight of $\bar{F}_{\mu\nu}^+$ is $c = -1$, thus showing the consistency of our assignment of chiral weights.

3. The coupling of one vector multiplet

The $N = 4$ superconformal multiplet contains, besides the gauge fields of the superconformal symmetries, a number of “matter” fields which are required to close the algebra. In table 1 we list the fields of the superconformal multiplet [15], indicating their properties and transformation character. Besides the independent fields of table 1, conformal supergravity contains a number of dependent gauge fields, which are determined by conventional constraints [23]. These are the fields ω_μ^{ab} , ϕ_μ^i and f_μ^a , corresponding to local Lorentz, S-supersymmetry and conformal boosts (K-transformations), respectively. The $N = 4$ constraints, and the properties of ω_μ^{ab} , ϕ_μ^i and f_μ^a are discussed extensively in [15]. The gauge field a_μ of the local $U(1)$ transformations is also dependent. It is the solution of

$$\phi^\alpha D_a \phi_\alpha = -\frac{1}{4} \bar{\Lambda}^i \gamma_a \Lambda_i. \quad (3.1)$$

The derivative $D_a = e_a^\mu D_\mu$ is covariant with respect to all superconformal symmetries. Note that ϕ_α is the only field of the superconformal multiplet which transforms under $SU(1,1)$.

TABLE 1
Fields of $N = 4$ conformal supergravity

Field	Type	Restrictions	SU(4)	w	c
ϕ_α	boson	$\phi^\alpha \phi_\alpha = 1$, $\phi^1 = \phi_1^*$, $\phi^2 = -\phi_2^*$	1	0	-1
Λ_i	fermion	$\gamma_5 \Lambda_i = \Lambda_i$	4	$\frac{1}{2}$	$-\frac{3}{2}$
E_{ij}	boson	$E_{ij} = E_{ji}$, complex	10	1	-1
$T_{ab}{}^{ij}$	boson	$T_{ab}{}^{ij} = -T_{ba}{}^{ij} = -T_{ab}{}^{ji}$, $\frac{1}{2} \varepsilon_{ab}{}^{cd} T_{cd}{}^{ij} = -T_{ab}{}^{ij}$	6	1	-1
$\chi^{ij}{}_k$	fermion	$\gamma_5 \chi^{ij}{}_k = \chi^{ij}{}_k$; $\chi^{ij}{}_k = -\chi^{ji}{}_k$; $\chi^{ij}{}_i = 0$	20	$\frac{3}{2}$	$-\frac{1}{2}$
$D^{ij}{}_{kl}$	boson	$D^{ij}{}_{kl} = \frac{1}{4} \varepsilon^{ijmn} \varepsilon_{klpq} D^{pq}{}_{mn}$; $D_{kl}{}^{ij} \equiv (D^{kl}{}_{ij})^* = D^{ij}{}_{kl}$; $D^{ij}{}_{kj} = 0$	20	2	0
e_μ^a	boson	vierbein	1	-1	0
ψ_μ^i	fermion	$\gamma_5 \psi_\mu^i = \psi_\mu^i$; gravitino	4	$-\frac{1}{2}$	$-\frac{1}{2}$
$V_{\mu i}^j$	boson	$V_{\mu i}^j \equiv (V_{\mu j}^i)^* = -V_{\mu i}^j$; $V_{\mu i}^i = 0$; SU(4) gauge field	15	0	0
b_μ	boson	dilatational gauge field	1	0	0

The $N = 4$ superconformal algebra takes the form [15]

$$\begin{aligned} [\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] &= \delta_G^{\text{cov}}(\xi^\mu) + \delta_{\text{Lorentz}}(\epsilon^{ab}) + \delta_Q(\epsilon_3^i) + \delta_S(\eta^i) + \delta_{\text{SU}(4)}(\Lambda_j^i) \\ &\quad + \delta_{\text{U}(1)}(\Lambda) + \delta_K(\Lambda_K^a), \end{aligned} \quad (3.2)$$

where δ_G^{cov} is a covariant general coordinate transformation [24]. Here we have

$$\begin{aligned} \xi^\mu &= 2\bar{\epsilon}_1^i \gamma^\mu \epsilon_{2i} + \text{h.c.}, \\ \epsilon_3^i &= \epsilon^{ijkl} \bar{\epsilon}_{1k} \epsilon_{2l} \Lambda_j, \\ \Lambda_j^i &= E^{ik} \epsilon_{klmj} \bar{\epsilon}_2^l \epsilon_1^m + \frac{1}{2} (\bar{\epsilon}_2^k \gamma^a \epsilon_{ij} + \text{h.c.}) \bar{\Lambda}^i \gamma_a \Lambda_k - \frac{1}{4} (\bar{\epsilon}_2^k \gamma^a \epsilon_{1k} + \text{h.c.}) \bar{\Lambda}^i \gamma_a \Lambda_j \\ &\quad - \frac{1}{4} (\bar{\epsilon}_2^i \gamma^a \epsilon_{ij} + \text{h.c.}) \bar{\Lambda}^k \gamma_a \Lambda_k - (\text{h.c.}; \text{traceless}). \end{aligned} \quad (3.3)$$

The parameters given in (3.3) are already sufficient to determine all modifications in the transformation rules of the Maxwell multiplet and we therefore refrain from giving ϵ^{ab} , η^i , Λ and Λ_K^a explicitly. The commutator of a Q- and an S-transformation is

$$[\delta_Q(\epsilon), \delta_S(\eta)] = \delta_D(\Lambda_D) + \delta_{\text{Lorentz}}(\epsilon^{ab}) + \delta_S(\eta_2^i) + \delta_{\text{SU}(4)}(\Lambda_j^i) + \delta_K(\Lambda_K^a), \quad (3.4)$$

with parameters

$$\begin{aligned} \Lambda_D &= -\bar{\eta}_i \epsilon^i + \text{h.c.}, \\ \epsilon^{ab} &= -2\bar{\eta}_i \sigma^{ab} \epsilon^i + \text{h.c.}, \\ \Lambda_j^i &= +2\bar{\eta}^i \epsilon_j - (\text{h.c.}; \text{traceless}), \\ \eta_2^i &= -\frac{1}{4} \epsilon^{ijkl} \bar{\eta}_k \gamma^a \epsilon_j \gamma_a \Lambda_l. \end{aligned}$$

The commutator (3.4) can be readily evaluated on all fields and provides useful checks.

The complete transformation rules which realize this algebra on the fields of the Maxwell multiplet are

$$\delta A_\mu = \Phi \left(\bar{\varepsilon}^i \gamma_\mu \psi_i - 2 \bar{\varepsilon}^i \psi_\mu{}^j \phi_{ij} + \bar{\varepsilon}_i \gamma_\mu \Lambda_j \phi^{ij} \right) + \text{h.c.}, \quad (3.5a)$$

$$\begin{aligned} \delta \psi_i = & -\sigma_{ab} \varepsilon_i \hat{\bar{F}}_{ab}^+ - 2 \mathcal{D} \phi_{ij} \varepsilon^j + E_{ij} \phi^{jk} \varepsilon_k \\ & + \tfrac{1}{2} \varepsilon_i \bar{\Lambda}_j \psi^j - \varepsilon_j \bar{\Lambda}_i \psi^j + \tfrac{1}{2} \gamma^a \varepsilon^k \bar{\Lambda}^l \gamma_a \Lambda_i \phi_{kl} - 2 \phi_{ij} \eta^j, \end{aligned} \quad (3.5b)$$

$$\delta \phi_{ij} = \bar{\varepsilon}_{[i} \psi_{j]} - \varepsilon_{ijk} \bar{\varepsilon}^k \psi^l. \quad (3.5c)$$

The scalar modifications, discussed in sect. 2, are conveniently expressed in terms of

$$\Phi = \phi^1 + \phi^2, \quad \Phi^* = \phi_1 - \phi_2. \quad (3.6)$$

$\hat{\bar{F}}_{ab}^+$ is the fully supercovariant form of the tensor \bar{F}_{ab}^+ (2.19c). Let us first introduce the supercovariant tensor \hat{F}_{ab}^+ , given by

$$\hat{F}_{ab}^+ = F_{ab}^+ + \Phi I_{ab}^+ + \Phi^* J_{ab}^+, \quad (3.7)$$

with

$$I_{ab}^+ = \tfrac{1}{2} \left(-\bar{\psi}_\mu{}^i \gamma^\mu \sigma_{ab} \psi_i + \bar{\psi}_{i\mu} \sigma_{ab} \gamma^\mu \Lambda_j \phi^{ij} + \left(2 \bar{\psi}_a{}^i \psi_b{}^j \phi_{ij} \right)^+ \right), \quad (3.8a)$$

$$J_{ab}^+ = \tfrac{1}{2} \left(\bar{\psi}_{i\mu} \sigma_{ab} \gamma^\mu \psi^i - \bar{\psi}^i_\mu \gamma^\mu \sigma_{ab} \Lambda^j \phi_{ij} + \left(2 \bar{\psi}_{ai} \psi_{bj} \phi^{ij} \right)^+ \right). \quad (3.8b)$$

We have found that

$$\hat{\bar{F}}_{ab}^+ = \frac{1}{\Phi} \hat{F}_{ab}^+ + \frac{\Phi^*}{\Phi} \left(-\tfrac{1}{2} \bar{\Lambda}^i \sigma_{ab} \psi_i + T_{abij} \phi^{ij} \right). \quad (3.9)$$

Thus the tensor H_{ab}^+ , defined in (2.19), is given by

$$\begin{aligned} H_{ab}^+ &= K_{ab}^+ + J_{ab}^+, \\ K_{ab}^+ &= -\tfrac{1}{2} \bar{\Lambda}^i \sigma_{ab} \psi_i + T_{abij} \phi^{ij}. \end{aligned} \quad (3.10)$$

We can write (3.9) in the form

$$\hat{\bar{F}}_{ab}^+ = \bar{F}_{ab}^+ + I_{ab}^+, \quad (3.11)$$

which shows explicitly that also $\hat{\bar{F}}_{ab}^+$ is SU(1,1) invariant.

The algebra contains, besides the terms contained in (3.2), gauge transformations of the vector field A_μ :

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = (3.2) + \delta_\Lambda(\Lambda = -4\bar{\Phi}\bar{\epsilon}_2^i\epsilon_1^j\phi_{ij} + \text{h.c.}), \quad (3.12)$$

and of course the spinor equation of motion when (3.12) is evaluated on ψ_i . One can also obtain the spinor equation of motion from (3.9). The supersymmetry variation of (3.9) must be SU(1,1) invariant. The variation contains a term proportional to Φ^*/Φ , which violates SU(1,1) invariance unless the term as a whole vanishes. This leads to the condition

$$\begin{aligned} \not{D}\psi_i + \tfrac{1}{2}\sigma \cdot \hat{F}^- \Lambda_i + \tfrac{1}{2}E_{ij}\psi^j - \tfrac{1}{2}\epsilon_{ijkl}\sigma \cdot T^{jk}\psi^l \\ - \chi_i^{kl}\phi_{kl} + \tfrac{1}{6}\phi_{ij}E^{jk}\Lambda_k - \tfrac{1}{3}\epsilon^{\alpha\beta}\phi_\alpha\not{D}\phi_\beta\phi_{ij}\Lambda^j - \tfrac{1}{8}\gamma_a\psi_j\bar{\Lambda}_i\gamma_a\Lambda^j = 0. \end{aligned} \quad (3.13)$$

The result agrees with the calculation of the commutator.

A supersymmetry variation of (3.13) will reveal the other equations of motion. We already know the equation for the vector field. It is given by (2.14), with $G_{\mu\nu}$ as in (2.19a) and (3.10). The scalar field equation is

$$\begin{aligned} \square\phi_{ij} - \left(T_{abij}\hat{F}_{ab}^+ - \tfrac{1}{2}\epsilon_{ijkl}T_{ab}^{kl}\hat{F}_{ab}^-\right) + \left(\bar{\chi}_{ij}^k\psi_k - \tfrac{1}{2}\epsilon_{ijkl}\bar{\chi}_m^{kl}\psi^m\right) + \tfrac{1}{2}D^{kl}{}_{ij}\phi_{kl} \\ - \tfrac{1}{6}\left(\bar{\psi}_{[i}\epsilon_{\alpha\beta}\phi^\alpha\not{D}\phi^\beta\Lambda_{j]} + \epsilon_{ijkl}\bar{\psi}^k\epsilon^{\alpha\beta}\phi_\alpha\not{D}\phi_\beta\Lambda^l\right) \\ - \tfrac{1}{12}\left(\bar{\Lambda}^k\psi_{[i}E_{j]k} - \epsilon_{ijkl}\bar{\Lambda}_m\psi^kE^{lm}\right) - \tfrac{1}{12}\phi_{ij}E^{kl}E_{kl} \\ - \tfrac{1}{3}\phi_{ij}\left(D_a\phi^\alpha\right)\left(D_a\phi_\alpha\right) + \tfrac{1}{12}\phi_{ij}\left(\bar{\Lambda}^k\not{D}\Lambda_k + \bar{\Lambda}_k\not{D}\Lambda^k\right) + \tfrac{1}{12}\phi_{ij}\bar{\Lambda}^k\Lambda^l\bar{\Lambda}_k\Lambda_l = 0. \end{aligned} \quad (3.14)$$

Here \square is the fully supercovariant d'Alembertian. Acting on ϕ_{ij} it takes the form

$$\square\phi_{ij} = e_a^\mu \left[\mathfrak{D}_\mu (D_a\phi_{ij}) + \tfrac{1}{4}\bar{\phi}_{\mu[i}\gamma_a\psi_{j]} - \tfrac{1}{4}\epsilon_{ijkl}\bar{\phi}_\mu^k\gamma_a\psi^l + f_\mu^a\phi_{ij} + \text{Q-covariantizations} \right]. \quad (3.15)$$

The derivative \mathfrak{D}_μ is covariant with respect to Lorentz, Weyl, SU(4) and U(1) symmetry, the S- and K-covariantizations are given explicitly.

From (3.13), (3.14) and (2.14) we can now construct the lagrangian density which yields these equations of motion. Of course, the fact that such a construction is possible is a crucial test for the consistency of the assumptions made thus far. We

give the result without further comment, and discuss its implications for Poincaré supergravity in the next section. The result is

$$\begin{aligned}
e^{-1}\mathcal{L} = & -\frac{1}{4}F_{\mu\nu}^+F^{\mu\nu} + \frac{1}{\Phi}(\phi^1 - \phi^2) - \frac{1}{2}\bar{\Psi}^i\gamma^\mu\mathcal{G}_\mu\Psi_i - \frac{1}{4}\mathcal{G}_\mu\phi_{ij}\mathcal{G}^\mu\phi^{ij} \\
& - \frac{1}{\Phi}\hat{F}_{\mu\nu}^+K^{\mu\nu} - \frac{1}{4}E^{kl}\bar{\Psi}_k\Psi_l + \frac{1}{4}\varepsilon^{ijkl}\bar{\Psi}_i\sigma\cdot T_{jk}\Psi_l + \bar{\Psi}_i\chi_{kl}^i\phi^{kl} \\
& + \frac{1}{8}\phi_{ij}\phi^{kl}D_{kl}^{ij} - \frac{1}{6}\bar{\Psi}_i\Lambda^k\phi^{ij}E_{jk} - \frac{1}{3}\bar{\Psi}_i\varepsilon_{\alpha\beta}\phi^\alpha\mathcal{D}\phi^\beta\Lambda_j\phi^{ij} \\
& + \frac{1}{16}\bar{\Psi}_i\gamma^a\Psi^j\bar{\Lambda}^i\gamma_a\Lambda_j - \frac{1}{2}\frac{\Phi^*}{\Phi}K_{\mu\nu}^+K^{\mu\nu} - \frac{1}{48}\phi_{ij}\phi^{ij}\left\{E^{kl}E_{kl}\right. \\
& + 4(D_a\phi^\alpha)(D_a\phi_\alpha) - (\bar{\Lambda}_k\mathcal{D}\Lambda^k + \bar{\Lambda}^k\mathcal{D}\Lambda_k) - \bar{\Lambda}^k\Lambda^l\bar{\Lambda}_k\Lambda_l\left.\right\} \\
& - \bar{\Psi}^i\gamma^\mu\phi_{\mu}^j\phi_{ij} + \frac{1}{4}f_{\mu}^{\mu}\phi_{ij}\phi^{ij} - \frac{1}{4}\bar{\Psi}_{\mu k}\sigma^{\mu\nu}\phi_{\nu}^k\phi_{ij}\phi^{ij} \\
& - \frac{1}{2}\bar{\Psi}^i\gamma^\mu\gamma^r\Psi_{\mu}^j(D_{\nu}\phi_{ij} + \mathcal{G}_{\nu}\phi_{ij}) - \frac{1}{2\Phi}(F_{\mu\nu}^+ + \hat{F}_{\mu\nu}^+)J^{\mu\nu} \\
& - \frac{1}{2}\phi_{ij}\phi^{kl}\bar{\Psi}_\mu\gamma^\mu\chi_{kl}^j - \frac{1}{2}\bar{\Psi}_i\gamma^\mu\Psi_{\mu}^jE^{ik}\phi_{jk} \\
& - \frac{1}{2}(\bar{\Psi}_j\Lambda^i)(\bar{\Psi}_i\gamma^\mu\Psi_{\mu}^j) + \frac{1}{4}\bar{\Psi}^i\gamma^\mu\gamma^a\Psi_{\mu}^k\bar{\Lambda}^l\gamma_a\Lambda_i\phi_{kl} - \frac{1}{48}\bar{\Psi}_{\mu k}\gamma^\mu\Lambda_i\phi_{ij}\phi^{ij}E^{kl} \\
& - \frac{1}{2}\bar{\Psi}_\mu^k\gamma^\mu\sigma_{ab}\psi_kT_{abij}\phi^{ij} + \frac{1}{4}\bar{\Psi}_\mu\sigma_{ab}\gamma^\mu\Lambda^lT_{ab}{}^{ij}\phi_{ij}\phi_{kl} \\
& - \frac{1}{24}\varepsilon_{\alpha\beta}\bar{\Psi}_\mu^k\gamma^\mu\phi^\alpha\mathcal{D}\phi^\beta\Lambda_k\phi_{ij}\phi^{ij} \\
& + \frac{1}{2}(\bar{\Psi}_{ak}\psi_{bi}\bar{\Lambda}_j\sigma^{ab}\psi^k + \bar{\Psi}_{ak}\sigma^{ab}\psi_{bi}\bar{\Lambda}_j\psi^k)\phi^{ij} + \frac{1}{2}\phi^{ij}\phi_{kl}T_{abij}\bar{\Psi}_a^k\psi_b{}^l \\
& - \frac{1}{2}\bar{\Psi}_{ai}\sigma^{ab}\psi_{bl}E_{jk}\phi^{ij}\phi^{kl} + \frac{1}{8}e^{-1}\varepsilon^{\mu\nu\lambda\rho}\phi_{kl}^i\phi_{kl}^j\bar{\Psi}_{i\mu}\gamma_\rho\psi_{\nu}^k\bar{\Lambda}^l\gamma_\lambda\Lambda_j \\
& + \frac{1}{4}e^{-1}\varepsilon^{\mu\nu\lambda\rho}\bar{\Psi}_{\mu k}\gamma_\nu\psi_{\lambda}^j\phi_{ij}^k\mathcal{G}_{\rho}^{\tilde{}}\phi_{ij} \\
& + \frac{1}{4}e^{-1}\varepsilon^{\mu\nu\lambda\rho}\bar{\Psi}_\mu^i\psi_{\nu}^j\bar{\Psi}_\nu^k\gamma_\lambda\psi_{\rho k}\phi_{ij} + \frac{1}{4}\bar{\Psi}_{\mu i}\psi_{\nu j}\bar{\Psi}_\nu^k\gamma^\mu\Lambda_i\phi^{ij}\phi^{kl} \\
& + \frac{1}{4}\bar{\Psi}_{\mu i}\psi_{\nu j}\bar{\Psi}_\nu^k\psi_{\rho}^l\phi^{ij}\phi_{kl} + \text{h.c.}
\end{aligned} \tag{3.16}$$

We have attempted to simplify the result somewhat by introducing covariant tensors $\hat{F}_{\mu\nu}^+$ and $D_{\mu}\phi_{ij}$ in suitable places. In particular, all terms of the form $(\psi_{\mu})^2(\psi)^2$ could be absorbed in this way. The tensors $J_{\mu\nu}^+$ and $K_{\mu\nu}^+$ are defined in (3.8b) and (3.10).

4. $N = 4$ Poincaré supergravity

The action (3.16) is quadratic and the transformation rules (3.5) are linear in the fields of the vector multiplet. Therefore we can generalize the results of sect. 3 to an arbitrary number of vector multiplets, say

$$A_\mu^I, \quad \psi_i^I, \quad \phi_{ij}^I, \quad I = 1, \dots, P, \quad (4.1)$$

which are minimally coupled, i.e. the action (3.16) is generalized to

$$e^{-1}\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^{+I}\eta_{IJ}F^{\mu\nu+J}\frac{1}{\Phi}(\phi^1 - \phi^2) + \dots, \quad (4.2)$$

where all terms are constructed in the same fashion, using a constant real metric η_{IJ} . Without loss of generality we can take η_{IJ} diagonal, with values ± 1 . The action (4.2) is invariant under global $\text{SO}(p, q)$ transformations, with $p + q = P$.

This generalization will be necessary when we consider the transition to the Poincaré theory. The $N = 4$ Poincaré supergravity multiplet contains 6 physical vector fields, so we expect that for $P < 6$ (4.2) does not lead to a consistent Poincaré theory. The η_{IJ} must be chosen such that canonical kinetic terms for all physical fields are obtained and ghosts are avoided.

The transition to the Poincaré gauge is made by imposing conditions which break the K-, D- and S-symmetries of the superconformal theory. To motivate these conditions, we consider the dependent field f_μ^a . Its trace, which appears in (3.16), is equal to

$$f_\mu^\mu = \frac{1}{6}R(\omega) + \frac{1}{6}\left\{\bar{\psi}_\mu^i\sigma^{\mu\nu}\phi_{\nu i} + \bar{\psi}_\mu^i T_{ij}^{\mu\nu}\psi_\nu^j + \text{h.c.}\right\}. \quad (4.3)$$

Here $R(\omega)$ is the Riemann scalar for the spin-connection ω_μ^{ab} , which contains the usual ψ_μ torsion. Thus we obtain the conventional action for the gravitational field if we choose in (4.2)

$$\phi_{ij}^I\eta_{IJ}\phi^{Jij} = -\frac{6}{\kappa^2}, \quad (4.4)$$

where $\kappa = (8\pi G)^{1/2}$, G is Newton's constant. We choose units such that $\kappa = 1$ henceforth. The condition (4.4) breaks Weyl invariance. To fix the S- and K-gauges we impose

$$\phi_{ij}^I\eta_{IJ}\psi^{Jj} = 0, \quad (4.5)$$

$$b_\mu = 0. \quad (4.6)$$

Before we consider the consequences of (4.4)–(4.6), let us go back to the action (4.2) (and (3.16)). It is linear in the fields D_{kl}^{ij} and χ_{ij}^k of the superconformal

multiplet, and the equations of motion of these fields impose further restrictions on ϕ_{ij}^I and ψ_i^I . The equations are

$$\phi_{ij}^I \eta_{IJ} \phi^{Jkl} = -\frac{1}{2} \delta_{[i}^k \delta_{j]}^l, \quad (4.7)$$

$$\phi_{ij}^I \eta_{IJ} \psi^{Jk} = 0, \quad (4.8)$$

where we have used (4.4), (4.5) to simplify.

Thus we must solve, in particular, (4.7) for the fields ϕ_{ij}^I . This is possible only if $P \geq 6$, and if at least 6 diagonal elements of η equal -1 . To see this, it is convenient to parametrize ϕ_{ij}^I in the form

$$\phi_{ij}^I(x) = x_m^I(x) \beta_{ij}^m + i x_{m+3}^I(x) \alpha_{ij}^m, \quad (4.9)$$

where α^m, β^m ($m = 1, 2, 3$) are real antisymmetric 4×4 matrices which generate $SU(2) \times SU(2)$. They satisfy

$$\begin{aligned} \{\alpha^m, \alpha^n\} &= \{\beta^m, \beta^n\} = -2\delta^{mn} \mathbb{1}, \\ [\alpha^m, \beta^n] &= 0, \\ [\alpha^m, \alpha^n] &= 2\epsilon^{mnp} \alpha^p, \quad [\beta^m, \beta^n] = 2\epsilon^{mnp} \beta^p. \end{aligned} \quad (4.10)$$

Explicit representations and useful properties of α^m and β^m can be found in the literature, e.g. [11]. The duality condition (2.1) for ϕ_{ij}^I implies that the $x_p^I(x)$ ($p = 1, \dots, 6$) are real. Thus we have 6 vectors in a real P -dimensional space with metric η . They must satisfy

$$x_p \cdot x_q = -\frac{1}{4} \delta_{pq}. \quad (4.11)$$

This immediately leads to the above conditions on η_{IJ} and P .

For $P = 6$ we obtain pure $N = 4$ Poincaré supergravity. The symmetries of the action are global $SO(6)$, where $SO(6)$ mixes the 6 compensating vector multiplets, and local $U(4)$, since the superconformal $SU(4)$ and $U(1)$ symmetries are still present. The scalar fields are constrained by (2.9) and (4.7). For $P = 6$ (4.8) implies that $\psi_i^I = 0$.

For $P > 6$ the additional vector multiplets must be coupled with $\eta_{IJ} = +1$, since only this choice gives the correct sign for the scalar kinetic terms. As we shall see the vector and spinor kinetic terms have the conventional form after elimination of auxiliary fields.

We now take $P > 6$ and otherwise arbitrary, and choose η_{IJ} as discussed above. We impose the conditions (4.4)–(4.6), and use the equations of motion (4.7), (4.8). The conditions (4.4)–(4.6) are not invariant under the transformation rules (3.5).

They are invariant only under a combination of superconformal Q-, S- and K-transformations, which then defines the Poincaré supersymmetry transformations:

$$\delta_Q^P(\varepsilon_i) = \delta_Q(\varepsilon_i) + \delta_S(\eta_i) + \delta_K(\Lambda_K^a), \quad (4.12)$$

with

$$\begin{aligned} \eta_i = \frac{1}{3} \Big\{ & -\sigma_{ab} \varepsilon^I \phi_{ij}^I \hat{F}_{ab}^{-J} - \frac{1}{2} \sigma_{ab} \varepsilon^I \bar{\psi}^{Ik} \sigma_{ab} \psi^J \varepsilon_{ijk l} \\ & - \frac{1}{2} \gamma^a \varepsilon_{[i} \bar{\psi}_{j]}^I \gamma_a \psi^{Jj} - 2 \phi_{ij}^I \mathcal{D} \phi^{Jjk} \varepsilon_k \Big\} \eta_{IJ} - \frac{1}{12} \gamma^a \varepsilon_{[i} \bar{\Lambda}_{j]} \gamma_a \Lambda^j, \\ \Lambda_K^a = & -\frac{1}{2} \bar{\varepsilon}^i \phi_{ai} + \text{h.c.} \end{aligned} \quad (4.13)$$

The Poincaré transformation rules of all fields can now be determined from (4.12), (4.13) and the known superconformal transformations ((3.5) and [15]).

The supersymmetry variation of (4.8) does not vanish, and the result must therefore be proportional to other equations of motion. Evaluating the Poincaré transformation (4.12) of (4.8) one finds that

$$E_{kl} = \bar{\psi}_k^I \psi_l^J \eta_{IJ}, \quad (4.14)$$

$$\left\{ \hat{F}_{ab}^+{}^I \phi_{ij}^J + \frac{1}{4} \bar{\psi}_{[i}^I \sigma_{ab} \psi_{j]}^J \right\} \eta_{IJ} = 0, \quad (4.15)$$

$$\phi^{Iik} \hat{\mathcal{D}}_{\mu}^J \phi_{jk}^I \eta_{IJ} = \bar{\psi}^{Ii} \gamma_{\mu} \psi_j^J \eta_{IJ} + \frac{1}{2} \bar{\Lambda}^i \gamma_{\mu} \Lambda_j - \text{traces}, \quad (4.16)$$

which correspond to the E_{kl} , T_{ab}^{ij} and $V_{\mu j}^i$ equations of motion, respectively. Of course, these equations can also be derived from the action (3.16) itself. To do this, one requires the explicit form of the dependent fields $\phi_{\mu i}$:

$$\sigma^{\mu\nu} \phi_{\nu i} = -\frac{1}{2} \varepsilon^{\mu\nu\lambda\rho} \gamma_{\nu} \mathcal{D}_{\lambda} \psi_{\rho i} + T^{\mu\nu} \psi_{ij}^j - \frac{1}{8} \varepsilon^{\mu\nu\lambda\rho} \varepsilon_{ijkl} \bar{\psi}_{\lambda}^j \psi_{\rho}^k \gamma_{\nu} \Lambda^l \quad (4.17)$$

and f_{μ}^{μ} , which was given in (4.3).

It is a straightforward calculation to eliminate the auxiliary fields E_{kl} and T_{ab}^{ij} . The kinetic terms in the resulting action read

$$\begin{aligned} e^{-1} \mathcal{L}_{\text{kin}} = & -\frac{1}{4} R(\omega) - \frac{1}{2} e^{-1} \varepsilon^{\mu\nu\lambda\rho} \bar{\psi}_{\mu}^I \gamma_{\nu} \mathcal{D}_{\lambda} \psi_{\rho i} - \frac{1}{4} \bar{\Lambda}^i \gamma^{\mu} \mathcal{D}_{\mu} \Lambda_i + \frac{1}{2} (D_{\mu} \phi^{\alpha}) (D^{\mu} \phi_{\alpha}) \\ & - \eta_{IJ} \left\{ + \frac{1}{4} \mathcal{D}_{\mu}^I \phi_{ij}^J \mathcal{D}^{\mu} \phi^{Iij} + \frac{1}{2} \bar{\psi}^{Ii} \gamma^{\mu} \mathcal{D}_{\mu} \psi_j^J + \frac{1}{4} F_{\mu\nu}^+{}^I F^{\mu\nu+J} \frac{1}{\Phi} (\phi^1 - \phi^2) \right\} \\ & - \frac{1}{2|\Phi|^2} \eta_{IJ} \eta_{KL} F_{\mu\nu}^+{}^I F^{\mu\nu+K} \phi_{ij}^J \phi^{Lij} + \text{h.c.} \end{aligned} \quad (4.18)$$

In this form the full action contains $V_{\mu j}^i$ and a_{μ} , and has local $\text{SU}(4) \times \text{U}(1)$

symmetry. The fields ϕ_{ij}^I and ϕ_α are still constrained. To write the action in terms of independent, unconstrained fields we must solve (2.9) and (4.7) explicitly, thus breaking $SU(4) \times U(1)$. A convenient choice for ϕ_α is

$$\phi_1 = \frac{1}{\sqrt{1-|Z|^2}}, \quad \phi_2 = \frac{Z}{\sqrt{1-|Z|^2}}. \quad (4.19)$$

In terms of Z the $U(1)$ gauge field is given by

$$a_\mu = + \frac{1}{2} \frac{Z^* \vec{\partial}_\mu Z}{1-|Z|^2} - \frac{1}{4} \bar{\Lambda}^i \gamma_\mu \Lambda_i, \quad (4.20)$$

and we find the conventional form of the scalar kinetic term [6]:

$$\frac{1}{2} D_\mu \phi^\alpha D^\mu \phi_\alpha + \text{h.c.} = - \frac{(\partial_\mu Z)(\partial^\mu Z^*)}{(1-|Z|^2)^2} + \psi_\mu^i, \Lambda_i \text{ dependent terms.} \quad (4.21)$$

The field Z parametrizes the manifold $SU(1,1)/U(1)$.

The scalar fields ϕ_{ij}^I also correspond to a non-linear σ -model, in this case $SO(n,6)/SO(n) \times SO(6)$, where n is the number of physical vector multiplets, i.e. $n = P - 6$. In the notation (4.9) it is clear how these groups act on ϕ_{ij}^I . The condition (4.11) is invariant under $SO(n,6)$ in an obvious way. The general solution of (4.11) can be written in the form

$$x_p^I = O^{IJ} x_p^{(0)J}, \quad (4.22)$$

where $x_p^{(0)}$ is a particular solution of (4.11), e.g.

$$x_p^{(0)J} = \begin{cases} \frac{1}{2} \delta_p^J & (J \leq 6), \\ 0 & (J > 6), \end{cases} \quad (4.23)$$

and $O^{IJ}(x)$ is an arbitrary element of $SO(n,6)$. The solution (4.23) is invariant under $SO(n)$, so that (4.22) contains effectively $6n + 15$ parameters. The action is invariant under local $SU(4)$, which acts as an $SO(6)$ rotation on the index p of x_p^I . Subtracting the 15 gauge degrees of freedom we find that the number of remaining real fields in ϕ_{ij}^I equals $6n$, as expected. For $n = 0$ all scalars ϕ_{ij}^I act as compensators for superconformal symmetries, and the solution (4.23) leads to the conventional action of pure $N = 4$ supergravity.

It is not possible to write, in the general case, the scalar fields ϕ_{ij}^I explicitly in terms of independent, unconstrained fields. For the case $n = 1$ an explicit solution

can be easily found. It is given by ($p, q = 1, \dots, 6$)

$$\begin{aligned} x_p{}^q &= \frac{1}{2} \left(\delta_{pq} + u_p u_q \frac{\cosh u - 1}{u^2} \right), \\ x_p{}^7 &= \frac{1}{2} u_p \frac{\sinh u}{u}, \quad u^2 = u_p u_p, \end{aligned} \quad (4.24)$$

where the $u_p(x)$ are six real fields.

Let us finally eliminate $V_{\mu j}^i$ from the kinetic term of ϕ_{ij}^I . Eq. (4.16) implies

$$V_{\mu j}^i = \frac{1}{2} \eta_{IJ} \left\{ \phi^{Iik} \vec{\partial}_\mu \phi_{jk}^J - \psi^{Ii} \gamma_\mu \psi_j^J \right\} - \frac{1}{4} \bar{\Lambda}^i \gamma_\mu \Lambda_j - \text{traces}, \quad (4.25)$$

and substitution of $V_{\mu j}^i$ gives

$$\begin{aligned} -\frac{1}{4} \eta_{IJ} \bar{\phi}_{\mu ij}^I \bar{\phi}^{\mu ij} \phi^{Jij} + \text{h.c.} &= -\frac{1}{2} \eta_{IJ} \left(\partial_\mu \phi_{ij}^I \right) \left(\partial^\mu \phi^{Jij} \right) + \frac{1}{4} \eta_{IJ} \eta_{KL} \phi^{Ijk} \vec{\partial}_\mu \phi_{ik}^J \phi^{Kil} \vec{\partial}^\mu \phi_{jl}^L \\ &\quad + \Lambda^i, \psi_i^I \text{ dependent terms.} \end{aligned} \quad (4.26)$$

This can be rewritten in terms of the real fields $x_p{}^I$:

$$e^{-1} \mathcal{L}_{\text{kin}, \phi} = -2 \eta_{IJ} \left(\partial_\mu x_p^I \right) \left(\partial^\mu x_p^J \right) - \eta_{IJ} \eta_{KL} \left(x_p^I \vec{\partial}_\mu x_q^J \right) \left(x_p^K \vec{\partial}^\mu x_q^L \right). \quad (4.27)$$

The kinetic term of the vector fields in (4.18) also has the canonical signs due to the last term in (4.18), which arises from the elimination of the auxiliary field T_{ab}^{ij} .

The symmetry of the equations of motion, which was $\text{SU}(1,1)$ for the case of a single vector multiplet in a superconformal background, is now extended to $\text{SO}(n,6) \times \text{SU}(1,1)$. Here $\text{SO}(n,6)$ is in fact a symmetry of the action as well.

The role of the Weyl multiplet in $N = 4$ Poincaré supergravity was first discussed in [25]. Using off-shell counting arguments [26], it was argued that the fields ϕ_α and Λ_i should correspond to the physical spin-0 and spin- $\frac{1}{2}$ fields of the Poincaré theory, and that E_{ij} should be auxiliary. The present version of $N = 4$ Poincaré supergravity confirms this. Nevertheless, we are still far from a complete off-shell theory, since the compensating sector remains completely on-shell. The fields D^{ij}_{kl} and χ^{ij}_k play a special role, as they occur linearly in the action (3.16). To make sense of (3.16) and its generalization, we have constrained the scalar and spinor fields by (4.7), (4.8), and thus eliminated D^{ij}_{kl} and χ^{ij}_k from the theory. In a complete off-shell theory this procedure should be circumvented by having additional auxiliary fields which couple to D^{ij}_{kl} and χ^{ij}_k [27].

5. Outlook

Even without a complete tensor calculus, superconformal methods may simplify the construction of Poincaré supergravity theories and their coupling to matter, as we have illustrated in the case of $N = 4$ supergravity and $N = 4$ vector multiplets. In the present work we have only considered abelian vector multiplets. There is no potential for the scalar fields ϕ_a and ϕ_{ij} , and as further inspection of (3.16) shows, there is no gravitino mass term either. In the final version of the Poincaré theory there are terms bilinear in ψ_μ^i , but these couple only to derivatives of bosonic fields. Thus the investigation of the $N = 4$ super-Higgs effect will have to await more general $N = 4$ matter couplings.

First of all, the extension to the non-abelian multiplet should be considered. The generalization of (3.16) will contain terms which are more than quadratic in the matter fields. One loses the generalized duality symmetry which we employed to our advantage in sect. 2, and the calculations will thus be more complicated than in the present case. Secondly, one can envisage the possibility of non-minimal coupling. No special cases of such couplings are known, and it is not at all clear that this can actually be done. In the truncations of $N = 8$ supergravity to $N = 4$ one finds six vector multiplets coupled to supergravity (and to spin- $\frac{3}{2}$ multiplets). There the symmetry of the equations of motion is $SO(6, 6) \times SU(1, 1)$ [9], which is the same as in the case of minimal coupling presented here. Presumably then, this truncation corresponds to minimal coupling as well. Nevertheless, in the absence of a complete tensor calculus it is difficult to exclude the possibility of non-minimal coupling. These matters are presently under investigation.

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